# LEVEL ONE HYPERASYMPTOTICS FOR THE FIRST PAINLEVE EQUATION (Recent Trends in Exponential Asymptotics)



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# Level one hyperasymptotics for the first Painlevé equation

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Abstract. In this paper we illustrate what is needed to construct the level one hyperasymptotic expansions for the first Painlevé equation. These level one hyperasymtotic expansions determine the solutions uniquely. Some details are given about solutions that are real-valued on the positive real axis.

## 1. Introduction and summary

In this paper we show what is needed to construct a level one hyperasymptotic expansion for special solutions of the first Painlevé equation. More details on hyperasymptotics for nonlinear ordinary differential equations (ODEs) are given in [6] and [7].

The main reason that we want to obtain hyperasymptotic expansions is that these expansions determine the solutions uniquely. In the case of the first Painlevé equation, the only solutions which are determined uniquely via their asymptotic expansions are the so-called tritronquée solutions. See [3]. As in [6,7] we will construct so-called transseries expansions. These transseries representations of solutions of the nonlinear ODE incorporate a free constant $C_j$, and when we cross a Stokes line in the transseries representation this constant changes its value to $C_j+K_j$. This is the Stokes phenomenon for nonlinear ODEs and more details are given in [6,7]. We will use the growth of the coefficients in the divergent asymptotic expansions to compute the Stokes multipliers $K_j$.

We are going to determine uniquely all solutions that are real-valued on the positive real axis via their level one hyperasymptotic expansion. Stokes multipliers play an important role in these results. In the literature these types of solutions are usually defined via medianization or balanced averaging of transseries. See for example [2]. These methods are not very practical. From our results it is obvious that these types of solutions form a one real-parameter family of solutions.

The set-up of this paper is as follows. In the first step we determine the transseries expansions in two directions of the complex plane. The transseries are convergent series of divergent series

$$u(z) \sim \sum_{n=0}^{\infty} C^n \tilde{u}_n(z). \tag{1.1}$$

Each of these formal series $\tilde{u}_n(z)$ can be resummed, and these sums $u_n(z)$ have again transseries expansions in both complex directions. This process of resumming of divergent series and expansion of the sums in transseries is purely formal and can be repeated.

In the second step we introduce real solutions $u_n(z)$ which are the Borel-Lapace transforms of the formal divergent series $\tilde{u}_n(z)$ in half-planes of the complex plane. In these half-planes the transseries expansion of $u_n(z)$ is just its divergent expansion, that is, the corresponding constant $C$ is zero. We will also need the transseries expansions of $u_n(z)$ in the two adjacent sectors. In these adjacent sectors

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the constant $C$ is equal to one of the Stokes multipliers. These three transseries expansions for $u_n(z)$ are the connection relations for this function. From these connection relations we obtain asymptotic formulae for the late coefficients in the divergent expansion $\tilde{u}_n(z)$. These asymptotic formulae for the late coefficients involve the Stokes multipliers, and since we know how to compute all the coefficients, we use these asymptotic formulae for the late coefficients to compute the Stokes multipliers. In general, it will only be possible to compute these Stokes multipliers numerically.

The connection relations and the Stokes multipliers is all the information that is needed to construct the hyperasymptotic expansions. In [4] the optimal number of terms at each level and estimates for the remainders are given. We will use these results and give the first two levels of the hyperasymptotic expansions. Finally, we also show why the Borel-Laplace transforms of the dominant asymptotic expansions are not real-valued on the positive real axis. The transseries expansions that correspond to the real-valued solutions have constants $C = A - \frac{1}{2}K$, where $K$ is the relevant Stokes multiplier, and $A$ is a free real parameter. The level one hyperasymptotic expansions determine these solutions uniquely.

In the case of the first Painlevé equation, the Borel-Laplace transforms of the dominant asymptotic expansions are the so-called tritronquée solutions. As mentioned in the previous paragraph these functions are not real-valued on the positive real axis. The positive real axis is an active Stokes line. However, these tritronquée solutions are real valued on one of the imaginary axis, which is an inactive anti-Stokes line. For more details on tritronquée solutions see [3].

## 2. The first Painlevé equation

The first Painlevé differential equation

$$\frac{d^2y}{dx^2} = 6y^2 + x \tag{2.1}$$

has solutions such that $y(x) \sim \pm i\sqrt{x/6}$ as $|x| \to \infty$. From an asymptotics point of view it makes sense to use the following transformation suggested by Boutroux [1]

$$y(x) = i\sqrt{x/6}\,u(z), \qquad \text{where } z = -\tfrac{48}{5}\left(-\tfrac{1}{6}x\right)^{5/4}, \tag{2.2}$$

which converts (2.1) to

$$u'' + \frac{u'}{z} - \tfrac{3}{2}\left(u^2 - 1\right) - \frac{4u}{25z^2} = 0. \tag{2.3}$$

This equation has solutions with asymptotic expansions

$$u(z) \sim \sum_{s=0}^{\infty} a_s z^{-s} \tag{2.4}$$

as $|z| \to \infty$, where the coefficients are defined via

$$\begin{gathered} a_0^2 = 1, \qquad a_2 = -\tfrac{4}{75}, \qquad a_0a_4 = -\tfrac{392}{5625}, \qquad a_{2m+1} = 0, \qquad m = 0, 1, 2, \cdots, \\ 3a_0a_{2m} = 4(m-1)^2 a_{2m-2} - \tfrac{3}{2}\sum_{p=2}^{m-2} a_{2p}a_{2m-2p}, \qquad m = 2, 3, 4, \cdots. \end{gathered} \tag{2.5}$$

There are no free parameters in these coefficients. The solutions having asymptotic expansion (2.4) have one free parameter which will show up in the transseries.

Since all the odd coefficients in (2.4) are zero we could have written (2.4) as an asymptotic expansion in powers of $z^{-2}$. However, the other asymptotic expansions in the transseries expansions are in powers of $z^{-1}$, and it is more convenient below to do the same in (2.4).

### 3. Transseries expansions

We substitute the transseries expansion

$$u(z) \sim \sum_{n=0}^{\infty} C_1^n \tilde{u}_n(z), \qquad \Re z \to +\infty, \tag{3.1}$$

into the nonlinear ode (2.3), and denote the Borel-Laplace transform of $\tilde{u}_n(z)$ as $u_n(z)$. The result is that $u_0(z)$ is, of course, a solution of the original nonlinear ode (2.3) and all the other $u_n(z)$ satisfy the linear inhomogeneous ode

$$u_n'' + \frac{u_n'}{z} - \left(3u_0 + \frac{4}{25z^2}\right) u_n = \tfrac{3}{2} \sum_{p=1}^{n-1} u_p u_{n-p}. \tag{3.2}$$

In (3.1) the $\tilde{u}_n(z)$ are formal divergent series and from (3.2) we obtain that

$$\tilde{u}_n(z) = e^{-n\sqrt{3}z} \sum_{s=0}^{\infty} a_{sn} z^{-s-(n/2)}, \tag{3.3}$$

where the coefficients $a_{s0}$ are given in (2.5). We will now take

$$a_0 = 1. \tag{3.4}$$

Since our main goal is a level one hyperasymptotic expansion we determine only the coefficients of $\tilde{u}_1(z)$. These coefficients are

$$a_{01} = 1, \qquad 2\sqrt{3} s a_{s1} = -\left(s - \tfrac{1}{2}\right)^2 a_{s-1,1} + 3 \sum_{p=4}^{s+1} a_{p0} a_{s-p+1,1}. \tag{3.5}$$

The only freedom that we have is the choice of $a_{01}$. We set it to unity and put that freedom in constant $C_1$.

We will also need the transseries expansion

$$u(z) \sim \sum_{n=0}^{\infty} C_2^n \tilde{v}_n(z), \qquad \Re z \to -\infty. \tag{3.6}$$

When we substitute this expansion into (2.3), and denote the Borel-Laplace transform of $\tilde{v}_n(z)$ as $v_n(z)$, we obtain that $v_0(z)$ is a solution of (2.3), for $n \geqslant 1$ the $v_n(z)$ satisfy (3.2), and for $\tilde{v}_n(z)$ we have the expansions

$$\tilde{v}_n(z) = e^{n\sqrt{3}z} \sum_{s=0}^{\infty} (-1)^s a_{sn} z^{-s-(n/2)}. \tag{3.7}$$

Each of the $u_1(z)$ and $v_1(z)$ have their own transseries expansions. In the 'easy' directions they are

$$u_1(z) \sim \sum_{p=1}^{\infty} p C_1^{p-1} \tilde{u}_p(z), \qquad \Re z \to +\infty, \tag{3.8a}$$

$$v_1(z) \sim \sum_{p=1}^{\infty} p C_2^{p-1} \tilde{v}_p(z), \qquad \Re z \to -\infty. \tag{3.8b}$$

These functions have also transseries expansions in the opposite directions, and these transseries expansions can be computed by simple substitution of the transseries into the differential equations (3.2). However, it seems that the structure of these transseries expansions is not as simple as (3.8).

Similarly, we can obtain a transseries expansion

$$v_1(z) \sim \sum_{p=-1}^{\infty} C_1^{p+1} \tilde{\mathrm{u}}_p(z), \qquad \Re z \to +\infty, \tag{3.9}$$

where $\tilde{\mathrm{u}}_{-1}(z) = \tilde{v}_1(z)$ and $\tilde{\mathrm{u}}_0(z) = \tilde{\mathrm{v}}_0(z)$.

At this moment we have all the information that is needed:

(1) We have the original expansion (2.4), and can compute the coefficients via (2.5).

(2) The dominant re-expansions are $u_1(z)$ and $v_1(z)$, and we can compute their coefficients via (3.5).

(3) We have also shown that the level two re-expansion would involve $u_2(z)$ and $v_2(z)$ and $\mathrm{u}_0(z)$ and $\mathrm{v}_0(z)$, and, hence, the level two re-expansion is of order $\exp(-2\sqrt{3}|z|)$. This information is needed to determine the optimal number of terms in the level one re-expansion.

## 4. The Stokes multipliers

The Stokes multipliers are related to what a Borel transform (not introduced in this example) defined in a complex Borel plane via a convergent expansion near, say, $t = k\sqrt{3}$, sees at the singularity at $t = \ell\sqrt{3}$. In this notation $k, \ell$ are integers, and the relevant Stokes multiplier is a constant $K_{k\ell}$.

We denote with $u_{n,\pm}(z)$ the Borel-Laplace transform of formal asymptotic expansion $\tilde{u}_n(z)$ in the sector $0 < \pm \operatorname{ph} z < \pi$. Then $u_{n,+}(z)$ and $u_{n,-}(z)$ have the same asymptotic expansion in the sector $|\operatorname{ph} z| < \frac{1}{2}\pi$. It follows from transseries expansion (3.8$a$) that

$$u_{0,-}(z) = \sum_{p=0}^{\infty} K_{0,-1}^p u_{p,+}(z). \tag{4.1}$$

Similarly, we denote with $v_{n,\pm}(z)$ the Borel-Laplace transform of formal asymptotic expansion $\tilde{v}_n(z)$ in the sector $-\pi < \pm(\pi + \operatorname{ph} z) < 0$. Then $v_{n,+}(z)$ and $v_{n,-}(z)$ have the same asymptotic expansion in the sector $-\frac{3}{2}\pi < \operatorname{ph} z < -\frac{1}{2}\pi$. It follows from transseries expansion (3.8$b$) that

$$v_{0,-}(z) = \sum_{p=0}^{\infty} K_{0,1}^p v_{p,+}(z). \tag{4.2}$$

From connection relations (4.1) and (4.2) we obtain

$$a_p \sim \frac{K_{0,-1}}{2\pi i} \sum_{s=0}^{\infty} \frac{a_{s1}\Gamma\left(p-s-\frac{1}{2}\right)}{\left(\sqrt{3}\right)^{p-s-\frac{1}{2}}} + \frac{K_{0,1}}{2\pi i} \sum_{s=0}^{\infty} \frac{(-1)^s a_{s1}\Gamma\left(p-s-\frac{1}{2}\right)}{\left(-\sqrt{3}\right)^{p-s-\frac{1}{2}}}, \tag{4.3}$$

as $p \to \infty$. Since $a_{2p+1} = 0$ it follows that

$$K_{0,-1} = iK_{0,1}, \tag{4.4}$$

and hence

$$a_{2p} \sim \frac{K_{0,1}}{\pi} \sum_{s=0}^{\infty} \frac{a_{s1}\Gamma\left(2p-s-\frac{1}{2}\right)}{\left(\sqrt{3}\right)^{2p-s-\frac{1}{2}}}, \tag{4.5}$$

as $p \to \infty$. Taking $p = 32$ and 32 terms on the right-hand side of (4.5) gives us

$$K_{0,1} = -0.46960787703009925108 6. \tag{4.6}$$

In this approximation all the digits are correct. In [8] the Stokes multiplier is computed via the isomonodromic deformations of linear equations associated with them. The exact value is

$$K_{0,1} = -\left(\frac{2\sqrt{3}}{5\pi}\right)^{1/2}. \tag{4.7}$$

It follows from the connection relations (4.1) and (4.2) that the function $u_{0,-}(z) = v_{0,-}(z)$ has transseries expansions

$$\begin{aligned} u_{0,-}(z) &\sim \sum_{p=0}^{\infty} (K_{0,-1})^p \, \tilde{u}_p(z), && 0 < \operatorname{ph} z < \tfrac{1}{2}\pi, \\ &\sim \tilde{u}_0(z), && -\pi < \operatorname{ph} z < 0, \\ &\sim \sum_{p=0}^{\infty} (K_{0,1})^p \, \tilde{v}_p(z), && -\tfrac{3}{2}\pi < \operatorname{ph} z < -\pi, \end{aligned} \tag{4.8}$$

## 5. The hyperasymptotic expansions

The function $u_{0,-}(z)$ is the unique solution of (2.3) with the property that $u_{0,-}(z) \sim 1$ as $|z| \to \infty$ in the sector $-\frac{3}{2}\pi < \operatorname{ph} z < \frac{1}{2}\pi$. The optimal number of terms in its asymptotic expansion $\tilde{u}_0(z)$ is $\sqrt{3}|z| + \mathcal{O}(1)$. In this section $N$ is an integer such that $N - \sqrt{3}|z| = \mathcal{O}(1)$ as $z \to \infty$. We refer to [4] for the optimal number of terms at each level, and for remainder estimates.

**Level 0.** This level is just the optimal truncated version of the asymptotic expansion (2.4). We have

$$u_{0,-}(z) = \sum_{s=0}^{N-1} a_s z^{-s} + R^{(0)}(z), \tag{5.1}$$

where

$$R^{(0)}(z) = e^{-\sqrt{3}|z|} \, |z|^{1/2} \, \mathcal{O}(1), \tag{5.2}$$

as $z \to \infty$ in the sector $-\pi < \operatorname{ph} z < 0$.

**Level 1.** Now we re-expand the remainder. The only information that we need are the transseries expansions in (4.8).

$$\begin{aligned} u_{0,-}(z) = \sum_{s=0}^{2N-1} a_s z^{-s} &+ z^{1-2N} \frac{K_{0,-1}}{2\pi i} \sum_{s=0}^{N-1} a_{s1} F^{(1)}\left(z; \begin{matrix} 2N - s - \frac{1}{2} \\ -\sqrt{3} \end{matrix}\right) \\ &+ z^{1-2N} \frac{K_{0,1}}{2\pi i} \sum_{s=0}^{N-1} (-1)^s a_{s1} F^{(1)}\left(z; \begin{matrix} 2N - s - \frac{1}{2} \\ \sqrt{3} \end{matrix}\right) + R^{(1)}(z), \end{aligned} \tag{5.3}$$

where

$$R^{(1)}(z) = e^{-2\sqrt{3}|z|} \, |z| \, \mathcal{O}(1), \tag{5.4}$$

as $z \to \infty$ in the sector $-\pi < \operatorname{ph} z < 0$.

The level one expansion is in terms of the first hyperterminant, which can be computed via

$$F^{(1)}\left(z; \begin{matrix} M \\ \sigma \end{matrix}\right) = -e^{\sigma z} (-z)^{M-1} \Gamma(M) \Gamma(1 - M, \sigma z),$$

where $\Gamma(a, z)$ is one of the incomplete gamma functions. For more details on hyperterminants see [5].

## 6. Real solutions on the real line

So far, the main solutions of (2.3) are $u_{0,\pm}(z)$. From an asymptotics point of view these solutions are very special, since they are uniquely determined by their asymptotic behaviour in a large sector. All the coefficients in the Poincaré asymptotic expansions (2.4) are real. However, these two functions are not real on the positive real axis. This is a direct consequence of the Stokes phenomenon that takes place when crossing the positive real axis. To obtain the imaginary part of $u_{0,-}(z)$ on the positive real axis we use the level one hyperasymptotic expansion (5.3), and the fact that for $z>0$ we have the identities

$$\Im\left(F^{(1)}\left(z;\begin{matrix}2N-s-\frac{1}{2}\\-\sqrt{3}\end{matrix}\right)\right)=\pi e^{-\sqrt{3}z}z^{2N-s-3/2}\qquad \Re\left(F^{(1)}\left(z;\begin{matrix}2N-s-\frac{1}{2}\\\sqrt{3}\end{matrix}\right)\right)=0. \tag{6.1}$$

Since $K_{0,-1}/i$ and $K_{0,1}$ are real (see (4.4) and (4.6)) we have

$$\Im\left(u_{0,-}(z)\right)\sim\frac{K_{0,-1}}{2i}e^{-\sqrt{3}z}z^{-1/2}\sum_{s=0}^{\infty}a_{s1}z^{-s}, \tag{6.2}$$

as $z\to\infty$ along the positive real axis.

To obtain a solution that is real on the real axis we have to choose a special value for $C_1$ in

$$u(z,C_1)=\sum_{n=0}^{\infty}C_1^n u_{n,-}(z). \tag{6.3}$$

The value of $C_1$ that cancels the imaginary part of $u_{0,-}(z)$ is of course $C_1=A-\frac{1}{2}K_{0,-1}$, where $A$ is any real number. Thus the function

$$u(z,A-\frac{1}{2}K_{0,-1})=\sum_{n=0}^{\infty}\left(A-\tfrac{1}{2}K_{0,-1}\right)^n u_{n,-}(z)=\sum_{n=0}^{\infty}\left(A+\tfrac{1}{2}K_{0,-1}\right)^n u_{n,+}(z) \tag{6.4}$$

is real on the positive real axis. On the positive real axis the solution $u(z,A-\frac{1}{2}K_{0,-1})$ is uniquely determined by its level one hyperasymptotic expansion:

$$\begin{aligned}u(z,A-\frac{1}{2}K_{0,-1})=\sum_{s=0}^{2N-1}a_sz^{-s}&+z^{1-2N}\frac{K_{0,-1}}{2\pi i}\sum_{s=0}^{N-1}a_{s1}F^{(1)}\left(z;\begin{matrix}2N-s-\frac{1}{2}\\-\sqrt{3}\end{matrix}\right)\\&+z^{1-2N}\frac{K_{0,1}}{2\pi i}\sum_{s=0}^{N-1}(-1)^sa_{s1}F^{(1)}\left(z;\begin{matrix}2N-s-\frac{1}{2}\\\sqrt{3}\end{matrix}\right)\\&+\left(A-\tfrac{1}{2}K_{0,-1}\right)e^{-\sqrt{3}z}z^{-1/2}\sum_{s=0}^{N-1}a_{s1}z^{-s}+R^{(1)}(z),\end{aligned} \tag{6.5}$$

where $N-\sqrt{3}z=\mathcal{O}(1)$ and

$$R^{(1)}(z)=e^{-2\sqrt{3}z}zO(1),\qquad \text{as } z\to+\infty. \tag{6.6}$$

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